Rational numbers vs. Irrational numbers

by

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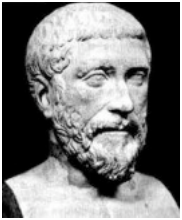
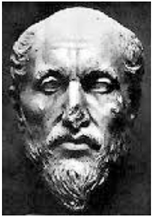
“The ultimate Nature of Reality is Numbers”

A quote from Pythagoras (570-495 BC)



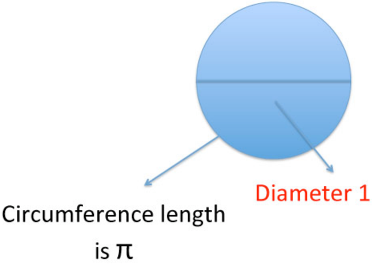
“Wherever there is number, there is beauty”

A quote from Proclus (412-485 AD)



Traditional Clock plus Circumference

$$1 \text{ min} = \frac{1}{60} \text{ of } 1 \text{ hour}$$

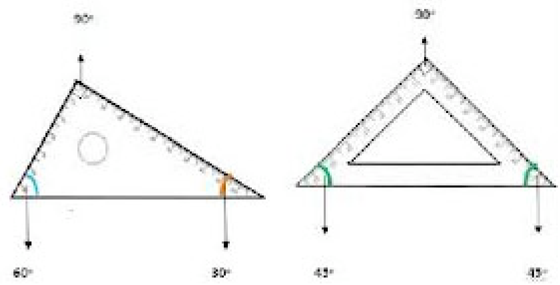


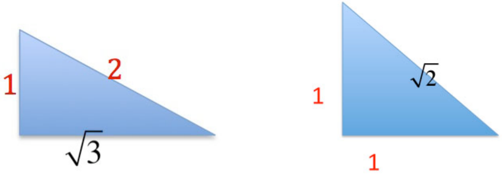
An Electronic Clock plus a Calendar

Hour : Minutes : Seconds
dd/mm/yyyy

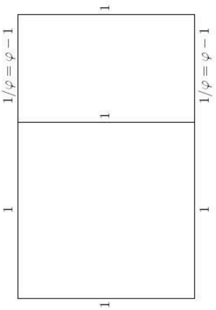
$$\begin{aligned}
1 \text{ month} &= \frac{1}{12} \text{ of } 1\text{year} \\
1 \text{ day} &= \frac{1}{365} \text{ of } 1 \text{ year (normally)} \\
1 \text{ hour} &= \frac{1}{24} \text{ of } 1 \text{ day} \\
1 \text{ min} &= \frac{1}{60} \text{ of } 1 \text{ hour} \\
1 \text{ sec} &= \frac{1}{60} \text{ of } 1 \text{ min}
\end{aligned}$$

TSquares: Use of Pythagoras Theorem

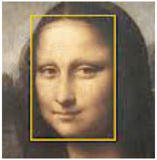




Golden number φ and Golden rectangle

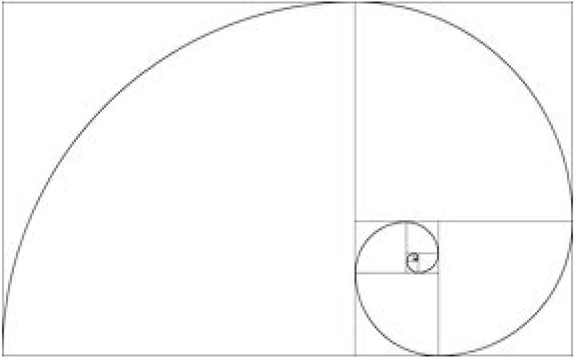


Roots of $x^2 - x - 1 = 0$ are $\varphi = \dfrac{1+\sqrt{5}}{2}$ and $-\dfrac{1}{\varphi} = \dfrac{1-\sqrt{5}}{2}$

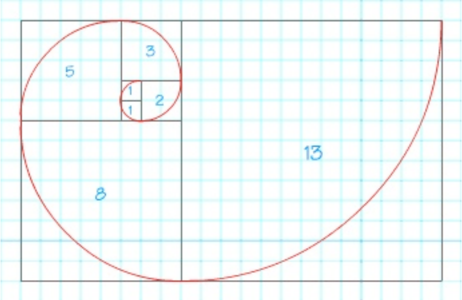


Golden number φ and Inner Golden spiral

Drawn with up to 10 golden rectangles



Outer Golden spiral and L. Fibonacci (1175-1250) sequence



$$\mathcal{F} = \{\underbrace{1}_{f_1}, \underbrace{1}_{f_2}, 2, 3, 5, 8, 13..., f_n, ...\} : \; f_n = f_{n-1} + f_{n-2}, \; n \geq 3$$

$$f_n = \frac{1}{\sqrt{5}}(\varphi^n + (-1)^{n-1}\frac{1}{\varphi^n})$$

Euler's Number e

$$s_3 = 1 + \frac{1}{1!} + \frac{1}{2} + \frac{1}{3!} = 2.6666....66....$$

$$s_4 = 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} = 2.70833333...333....$$

$$s_5 = 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = 2.7166666666...66....$$

....

$$\lim_{n\to\infty} \{1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + ... + \frac{1}{n!}\} = e = 2.718281828459........$$

e is an irrational number discovered by L. Euler (1707-1783), a limit of a sequence of rational numbers.

Definition of Rational and Irrational numbers

- A **Rational number** r is defined as:

$$r = \frac{m}{n}$$

  where m and n are integers with $n \neq 0$.

- Otherwise, if a number cannot be put in the form of a ratio of 2 integers, it is said to be an **Irrational number**.

Distinguishing between rational and irrational numbers

Any number x, (rational or irrational) can be written as:

$$x = I + f$$

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- I is its integral part;
- $0 \leq f < 1$ is its fractional part.

Examples

- $\frac{48}{25} = 1 + 0.92$
- $\frac{8}{3} =$
- $\frac{17}{7} =$
- $\sqrt{2} =$
- $\pi =$
- $\varphi = \frac{1+\sqrt{5}}{2} =$

Answers to Examples

- $\frac{48}{25} = 1 + 0.92$
- $\frac{8}{3} = 2 + 0.6666666.....$
- $\frac{17}{7} = 2 + 0.4285714285714.....$
- $\sqrt{2} = 1 + 0.4142135623731.....$
- $\pi = 3 + 0.14159265358979.....$
- $\varphi = 1 + 0.6180339887499...$

Distinguishing between rational and irrational numbers

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1. As $x = I + f$, I: Integer; $0 < f < 1$: Fractional.

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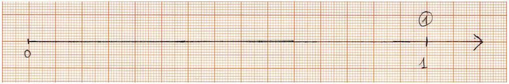
1. As $x = I + f$, I: Integer; $0 < f < 1$: Fractional.
2. $\Longrightarrow$ Distinction between rational and irrational can be restricted to fraction numbers f between $0 < f < 1$.

Position of the Problem

$$\mathcal{R} = \{\text{Rational Numbers } f,\ 0 < f < 1\}$$

$$\mathcal{I} = \{\text{Irrational Numbers } f,\ 0 < f < 1\}$$

The segment following segment $\mathcal{S}$ represents all numbers between 0 and 1:



$$\mathcal{S} = \mathcal{R} \cup \mathcal{I} \text{ with } \mathcal{R} \cap \mathcal{I} = \Phi \text{ empty set.}$$

- **Basic Question:**

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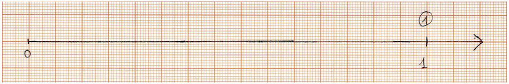

$$\mathcal{S} = \mathcal{R} \cup \mathcal{I} \text{ with } \mathcal{R} \cap \mathcal{I} = \Phi \text{ empty set.}$$

- **Basic Question:**
- If we pick a number f **at random** between 0 and 1, what is the probability that this number be rational: $f \in \mathcal{R}$?

The Decimal Representation of a number

Any number $f : 0 < f < 1$ has the following decimal representation:

$$f \overbrace{=}^{Notation} 0.d_1 d_2 d_3 ... d_k ...$$

$$d_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$f = d_1(\frac{1}{10}) + d_2(\frac{1}{100}) + d_3(\frac{1}{1000}) + ... + d_k(\frac{1}{10^k}) + ..$$

with at least one of the d_i's $\neq 0$.

Main Theorem about Rational Numbers

The number $0 < f < 1$ is rational, that is $f = \frac{m}{n}$, $m < n$,

<u>if and only if</u>

its decimal representation:

$$\begin{aligned} f &= 0.d_1d_2d_3...d_k... \\ &= d_1(\frac{1}{10}) + d_2(\frac{1}{10^2}) + d_3(\frac{1}{10^3}) + ... + d_k(\frac{1}{10^k}) + ... \end{aligned}$$

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takes one of the following forms:

f is either **Terminating**: $d_i = 0$ for $i > l \geq 1$
or f is **Non-Terminating** with a repeating pattern.

Proof of the Main Theorem about Rational Numbers

Theorem

The number $0 < f < 1$ *is rational, that is* $f = \frac{m}{n}$, $m < n$, **if and only if** *its decimal representation:*

$$f = 0.d_1 d_2 d_3 ... d_k ...$$

is either **Terminating** *(*$d_i = 0$ *for* $i > l \geq 1$*) or is* **Non-Terminating** *with a repeating pattern.*

Proof of the only if part of Main Theorem about Rational Numbers

Proof.

□

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1. If f has a terminating decimal representation, then f is rational.

□

Proof of the only if part of Main Theorem about Rational Numbers

Proof.

1. If f has a terminating decimal representation, then f is rational.
2. If f has a non-terminating decimal representation with a repeating pattern, then f is rational.

□

Proof of the first Statement of only if part

Statement 1: If f has a terminating decimal representation, then f is rational. **Consider:**

$$f \;=\; d_1(\frac{1}{10}) + d_2(\frac{1}{100}) + d_3(\frac{1}{1000}) + ... + d_k(\frac{1}{10^k})$$

then:

$$10^k f = d_1 10^{k-1} + d_2 10^{k-2} + ... + d_k.$$

implying:

$$f = \frac{m}{10^k} \text{ with } m = d_1 10^{k-1} + d_2 10^{k-2} + ... + d_k$$

Example

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$$0.625 = \frac{625}{1,000} = \frac{125 \times 5}{125 \times 8}$$

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$$0.625 = \frac{625}{1,000} = \frac{125 \times 5}{125 \times 8}$$

$$0.625 = \text{ after simplification: } \frac{5}{8}$$

Proof of the second Statement of only if part

Statement 2: If f has a non terminating decimal representation with repeating pattern, then f is rational.
Without loss of generality, consider:

$$f = 0.\overline{d_1 d_2 d_3 \ldots d_k} = 0.d_1 d_2 d_3 \ldots d_k d_1 d_2 d_3 \ldots d_k d_1 d_2 d_3 \ldots d_k \ldots$$

$$f = d_1(\frac{1}{10}) + d_2(\frac{1}{100}) + d_3(\frac{1}{1000}) + \ldots + d_k(\frac{1}{10^k}) + \frac{1}{10^k}[d_1(\frac{1}{10}) + d_2(\frac{1}{100}) + d_3(\frac{1}{1000}) + \ldots + d_k(\frac{1}{10^k})] + \frac{1}{10^{2k}}[.$$

then:

$$10^k f = \underbrace{d_1 10^{k-1} + d_2 10^{k-2} + \ldots + d_k}_{m:\, Integer} + f.$$

implying:

$$\underbrace{(10^k - 1)}_{n:\, Integer} f = m \iff f = \frac{m}{n}$$

Example on Proof of the second Statement

$$f = 0.\overline{428571} = 0.428571428571428571...$$

$$f = 4(\frac{1}{10})+2(\frac{1}{100})+8(\frac{1}{10^3})+5(\frac{1}{10^4})+7(\frac{1}{10^5})+1\frac{1}{10^6}+\frac{1}{10^6}(f)$$

$$10^6 \times f = 4\times 10^5+2\times 10^4+8\times 10^3+5\times 10^2+7\times 10+1+f$$

$$(10^6-1)\times f = 428,571$$

$$f = \frac{428,571}{10^6-1} = \frac{428,571}{999,999}$$

After simplification:

$$f = \frac{428,571}{999,999} = \frac{3\times 142,857}{7\times 142,857} = \frac{3}{7}$$

Proof of the "IF PART"

$$\boxed{f = 0.d_1 d_2 d_3 \ldots d_k \ldots \in \mathcal{R}}$$

$$\Downarrow$$

f has a terminating representation,

or

f has a non-terminating representation with a repeating pattern.

Tools for Proof of the if part of Main Theorem about Rational Numbers

Two tools to prove this result:

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1. **Euclidean Division Theorem**

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1. **Euclidean Division Theorem**
2. **Pigeon Hole Principle**

First Tool: Euclidean Division Theorem

$M \geq 0$ and $N \geq 1$.
Then, there exists a **unique pair** of integers (d, r), such that:

$$M = d \times N + r,$$

or equivalently:

$$\frac{M}{N} = d + \frac{r}{N}$$

$d \geq 0$ is the quotient of the division, and
$r \in \{0, 1, ..., N-1\}$ is the remainder.

Application of Euclidean Division Theorem on f, $0 < f < 1$

$$f = \frac{m}{n} = d_1(\frac{1}{10}) + d_2(\frac{1}{100}) + d_3(\frac{1}{1000}) + ... + d_k(\frac{1}{10^k}) + ...$$

$$\frac{10m}{n} = d_1 + f_1 \text{ where } f_1 = d_2(\frac{1}{10}) + d_3(\frac{1}{100}) + ... + d_k(\frac{1}{10^{k-1}}) + ...$$

$$10m = d_1 n + r_1 \qquad \frac{10m}{n} = d_1 + f_1 \qquad f_1 = \frac{r_1}{n} = d_2(\frac{1}{10}) + ...$$

$$10r_1 = d_2 n + r_2 \qquad \frac{10r_1}{n} = d_2 + f_2 \qquad f_2 = \frac{r_2}{n} = d_3(\frac{1}{10}) + ...$$

$$\vdots$$

$$10r_{k-1} = d_k n + r_k \qquad \frac{10r_{k-1}}{n} = d_k + f_k \qquad f_k = \frac{r_k}{n} = d_{k+1}(\frac{1}{10}) + ...$$

$$\vdots$$

Each of $r_1, r_2, ..., r_k, .. \in \{\underbrace{0}, \overbrace{1, ..., n-1}\}$

Application of Euclidean Division Theorem on f, $0 < f < 1$

$$f = \frac{m}{n} = d_1(\frac{1}{10}) + d_2(\frac{1}{100}) + d_3(\frac{1}{1000}) + ... + d_k(\frac{1}{10^k}) + ...$$

$\frac{10m}{n} = d_1 + f_1$ where $f_1 = d_2(\frac{1}{10}) + d_3(\frac{1}{100}) + ... + d_k(\frac{1}{10^{k-1}}) + ...$

$10m = d_1 n + r_1$	$\frac{10m}{n} = d_1 + f_1$	$f_1 = \frac{r_1}{n} = d_2(\frac{1}{10}) + ...$
$10r_1 = d_2 n + r_2$	$\frac{10r_1}{n} = d_2 + f_2$	$f_2 = \frac{r_2}{n} = d_3(\frac{1}{10}) + ...$
	$\vdots$	
$10r_{k-1} = d_k n + r_k$	$\frac{10r_{k-1}}{n} = d_k + f_k$	$f_k = \frac{r_k}{n} = d_{k+1}(\frac{1}{10}) + ...$
	$\vdots$	

Each of $r_1, r_2, ..., r_k, .. \in \{\underbrace{0}, \overbrace{1, ..., n-1}\}$

The Algorithm of Successive Multiplications by 10 and Divisions by n

- Can this procedure terminate?

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- yes, when $r_k = 0$.

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- Can this procedure terminate?
- yes, when $r_k = 0$.
- If not, $\{\boldsymbol{d_i}, \boldsymbol{r_i}\}$ starts repeating.

Proof of Terminating Sequences using Successive Multiplications and Divisions

$$\frac{10m}{n} = d_1 + d_2(\frac{1}{10}) + d_3(\frac{1}{100}) + ... + d_k(\frac{1}{10^{k-1}}) + ...$$

$10m = d_1 n + r_1$	$\frac{10m}{n} = d_1 + f_1$	$f_1 = \frac{r_1}{n} = d_1 + d_2(\frac{1}{10}) + ...$
$10r_1 = d_2 n + r_2$	$\frac{10r_1}{n} = d_2 + f_2$	$f_2 = \frac{r_2}{n} = d_2 + d_3(\frac{1}{10}) + ...$
		$\vdots$
$10r_{k-1} = d_k n + 0$	$\frac{10r_{k-1}}{n} = d_k + f_k$	$f_k = 0$

Algorithm stops at $k: \; r_k = 0$ implies:

$$r_{k+1} = r_{k+2} = ... = 0 \text{ and } d_{k+1} = d_{k+2} = ... = 0.$$

$$\Longrightarrow \frac{m}{n} = 0.d_1 d_2 .... d_k.$$

Examples of fractions with terminating decimal representation

1. $\frac{m}{n} = \frac{1}{4}$, $m = 1$, $n = 4$

$$10 \times 1 = 2 \times 4 + 2 \quad \Leftrightarrow \quad \frac{10\times 1}{4} = 2 + \frac{2}{4},\ (d_1 = 2,\ r_1 = 2)$$

$$10 \times 2 = 5 \times 4 + 0 \quad \Leftrightarrow \quad \frac{10\times 2}{4} = 5 + \frac{0}{4},\ (d_2 = 5,\ r_2 = 0)$$

$r_2 = 0$ implies $\frac{1}{4} = 0.d_1 d_2 = 0.25$

Examples of fractions with terminating decimal representation

2. $\frac{m}{n} = \frac{5}{8}$, $m = 5$, $n = 8$

Examples of fractions with terminating decimal representation

2. $\frac{m}{n} = \frac{5}{8}$, $m = 5$, $n = 8$

$$10 \times 5 = 6 \times 8 + 2 \quad \Leftrightarrow \quad \frac{10\times 5}{8} = 6 + \frac{2}{8}, \ (d_1 = 6, \ r_1 = 2)$$

$$10 \times 2 = 2 \times 8 + 4 \quad \Leftrightarrow \quad \frac{10\times 2}{8} = 2 + \frac{4}{8}, \ (d_2 = 2, \ r_2 = 4)$$

$$10 \times 4 = 5 \times 8 + 0 \quad \Leftrightarrow \quad \frac{10\times 4}{8} = 5 + \frac{0}{8}, \ (d_3 = 5, \ r_3 = 0)$$

$r_3 = 0$ implies $\frac{5}{8} = 0.d_1 d_2 d_3 = 0.625$

Successive Multiplications and Divisions: Non Terminating Representations

$$\frac{10m}{n} = d_1 + d_2(\frac{1}{10}) + d_3(\frac{1}{100}) + ... + d_k(\frac{1}{10^{k-1}}) + ...$$

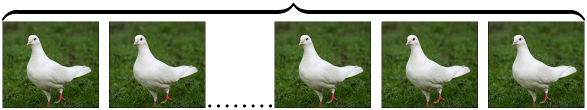
$$\begin{aligned}
10m &= d_1 n + r_1 &\Leftrightarrow&\quad \tfrac{10m}{n} = d_1 + \tfrac{r_1}{n} = d_1 + d_2(\tfrac{1}{10}) + ... \\
10r_1 &= d_2 n + r_2 &\Leftrightarrow&\quad \tfrac{10r_1}{n} = d_2 + \tfrac{r_2}{n} = d_2 + d_3(\tfrac{1}{10}) + ... \\
& &\vdots& \\
10r_{k-1} &= d_k n + r_k &\Leftrightarrow&\quad \tfrac{10r_{k-1}}{n} = d_k + \tfrac{r_k}{n} = d_k + d_{k+1}(\tfrac{1}{10}) + ... \\
& &\vdots&
\end{aligned}$$

Each of $r_1, r_2, ..., r_k, .. \in \{\overbrace{1, ..., n-1}\}$ and $r_i \neq 0$ for all i.

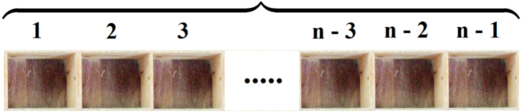
Second tool: Use of Pigeon hole Principle in proving that Infinite representations for $\frac{m}{n}$ have repeating patterns

Statement:

If you have n pigeons



to occupy $n - 1$ holes:

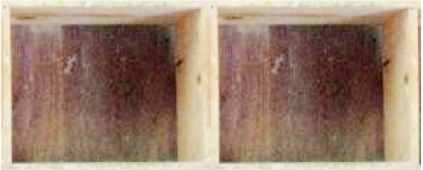


Then **at least 2 pigeons must occupy the same hole.**

Example 10 pigeons and 9 pigeon holes

Example of 3 pigeons and 2 pigeon holes





Solution of example of 3 pigeons and 2 pigeon holes

OR

Application of Pigeonhole Principle for non-terminating sequences

$$
\begin{array}{lcl}
10m = d_1 n + r_1 & \Leftrightarrow & \frac{10m}{n} = d_1 + \frac{r_1}{n} \\
10r_1 = d_2 n + r_2 & \Leftrightarrow & \frac{10r_1}{n} = d_2 + \frac{r_2}{n} \\
 & \vdots & \\
10r_{k-1} = d_k n + r_k & \Leftrightarrow & \frac{10r_{k-1}}{n} = d_k + \frac{r_k}{n} \\
 & \vdots &
\end{array}
$$

$r_1 =$  $r_2 =$  .... $r_{n-1} =$  $r_n =$ 

1 **2** **3** **n - 3** **n - 2** **n - 1**

By Pigeonhole principle: At least 2 remainders r_j, r_k, $1 \leq j < k \leq n$: $r_j = r_k$.

Applying the Pigeon hole Principle to obtain repeating sequences

Let $\{j, k\}$ be the first pair, such that:
$1 \leq j < k \leq n$ and $r_j = r_k$ then:

$$10r_j = d_{j+1}n + r_{j+1} \text{ and } 10r_k = d_{k+1}n + r_{k+1}$$

$$\Downarrow$$

$$d_{j+1} = d_{k+1} \text{ and } r_{j+1} = r_{k+1}...$$

More generally,

$$d_{j+l} = d_{k+l} \text{ and } r_{j+l} = r_{k+l},\ 1 \leq l \leq k - j.$$

and therefore by recurrence:

$$\frac{m}{n} = 0.d_1d_2...d_j\overline{d_{j+1}....d_k}$$

Example

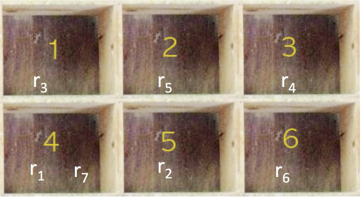
$$f = \frac{m}{n} = \frac{6}{7}$$

$$\begin{array}{ll}
10 \times 6 = 8 \times 7 + 4 & d_1 = 8 \; r_1 = 4 \\
10 \times 4 = 5 \times 7 + 5 & d_2 = 5 \; r_2 = 5 \\
10 \times 5 = 7 \times 7 + 1 & d_3 = 7 \; r_3 = 1 \\
10 \times 1 = 1 \times 7 + 3 & d_4 = 1 \; r_4 = 3 \\
10 \times 3 = 4 \times 7 + 2 & d_5 = 4 \; r_5 = 2 \\
10 \times 2 = 2 \times 7 + 6 & d_6 = 2 \; r_6 = 6 \\
10 \times 6 = 8 \times 7 + 4 & d_7 = 8 \; r_7 = 4 \\
 & \vdots
\end{array}$$

Each of $r_1, r_2, r_3, r_4, r_5, \ldots \in \{\overbrace{1, 2, 3, 4, 5, 6}\}$.

Example $f = \frac{m}{n} = \frac{6}{7}$

$r_1 = 4$ $r_2 = 5$ $r_3 = 1$ $r_4 = 3$ $r_5 = 2$ $r_6 = 6$ $r_7 = 4$



$\{1, 7\}$ is the first pair, such that $r_1 = r_7$ then:

$$\frac{6}{7} = 0.d_1\overline{d_2 d_3 d_4 d_5 d_6 d_7} = 0.8\overline{571428}$$

Length of pattern is 6.

Exercise

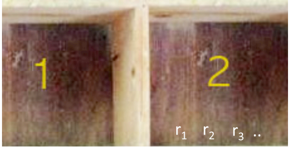
Find the decimal representation of

$$f = \frac{m}{n} = \frac{2}{3}$$

using Successive Multiplications and Divisions

Solution of the exercise $f = \frac{m}{n} = \frac{2}{3}$

$$\begin{array}{ll} 10 \times 2 = 6 \times 3 + 2 & d_1 = 6 \; r_1 = 2 \\ 10 \times 2 = 6 \times 3 + 2 & d_2 = 6 \; r_2 = 2 \\ 10 \times 2 = 6 \times 3 + 2 & d_3 = 6 \; r_3 = 2 \\ & \vdots \end{array}$$



$\{1, 2\}$ is the first pair, such that $r_1 = r_2$ and therefore:

$$\frac{2}{3} = 0.d_1\overline{d_2} = 0.6\overline{6}$$

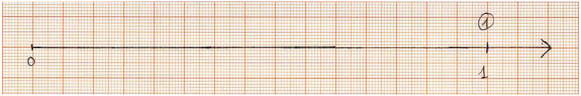
Length of pattern is 1

Answer to the Main question of Module

$$\mathcal{R} = \{\text{Rational Numbers } f,\ 0 < f < 1\}$$

$$\mathcal{I} = \{\text{Irrational Numbers } f,\ 0 < f < 1\}$$

$$\mathcal{S} = \mathcal{R} \cup \mathcal{I} \ \text{ with } \mathcal{R} \cap \mathcal{I} = \Phi \text{ empty set.}$$



Question: If we pick at random a number f between 0 and 1, what is the probability that this number be rational: $f \in \mathcal{R}$?

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- While if $f \in \mathcal{I}: \ f = 0.d_1d_2..d_k....$ (infinite representation with no specific pattern).
- Hence, "much more" ways to obtain elements in $\mathcal{I}$ than in $\mathcal{R}$.

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- To understand this concept, define for $n = 1, 2, 3, 4, ...$:

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- Examples of $\mathcal{R}_n$:
  $n = 1 : \mathcal{R}_1 = \{\frac{1}{2}\} = \{r_1\}$
  $n = 2 : \mathcal{R}_2 = \{\frac{1}{3}, \frac{2}{3}\} = \{r_2, r_3\}$
  $n = 3 : \mathcal{R}_3 = \{\frac{1}{4}, \frac{3}{4}\} = \{r_4, r_5\}$
  $n = 4 : \mathcal{R}_4 = \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\} = \{r_6, r_7, r_8, r_9\}$

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- Check $n = 5 : \mathcal{R}_5 = \{\frac{1}{6}, ?\}$

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- $\mathcal{R}_5 = \{\frac{1}{6}, \frac{5}{6}\} = \{r_{10}, r_{11}\}$

- As a consequence, we can **enumerate** the elements of $\mathcal{R}$:

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- Implying:
  Countable infinity of $\mathcal{R} \Longleftrightarrow$ a one to one relation between $\mathcal{R}$ and the natural integers: $\mathbb{N} = \{1, 2, 3, 4...\}$

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- This follows from the fact that f is irrational **if and only if** its infinite representation $0.d_1d_2...d_k...$ has all its elements belonging **randomly** to the set $\{0, 1, 2, ...9\}$.
- At that point, the proof of uncountability of $\mathcal{I}$ can be obtained using Cantor's proof by contradiction.

- Let us assume "countability of $\mathcal{I}$", i.e. its elements can be listed as $\{i_1, i_2, i_3, ...\}$, a set in a one-one relation with the set of natural numbers.

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  $i_2 = 0.f_{2,1}f_{2,2}...f_{2,k}....$
  ....
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- Let $\bar{i} = 0.\overline{f}_{1,1}, \overline{f}_{2,2}, ..., \overline{f}_{k,k}....$, such that the $\{\overline{f}_{i,i}\}$'s are randomly chosen with: $\overline{f}_{1,1} \neq f_{1,1}$, $\overline{f}_{2,2} \neq f_{2,2}, ..., \overline{f}_{k,k} \neq f_{k,k}, ...$

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- Contradiction: $\bar{i} \in \mathcal{I}$ but $\bar{i}$ different from each of the elements in $\{i_1, i_2, i_3...\}$.

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$$\boxed{\implies \text{Prob}(f \in \mathcal{R}) = \frac{\aleph_0}{\aleph_0 + \mathcal{C}} \approx \frac{\aleph_0}{\mathcal{C}} \approx 0.}$$