Flaws of Averages

BLOSSOMS Module

Supplement to Flaw of Averages #3:

"The average depends on your perspective"

Contents:

Rhonda's Dance Class

For Rhonda's Dance Class example, she mentions that she has 45 dancers in her beginner's class and 5 dancers in her advanced class.

From Rhonda's perspective, the average number of dancers is simply

$$\frac{45+5}{2} = 25.$$

From the dancers' perspective, if you asked all 50 of the dancers 'How many dancers are in your class', you would have 45 dancers respond with '45' and 5 dancers respond with '5'. The average of these 50 responses is

$$\frac{45*45+5*5}{50} = 41,$$

which is clearly different than 25. Hence, 'the average depends on your perspective'.

Something that we have found helpful is consolidating the results into a table such as this:

The Average…	From the Perspective of…	Yields a Value of…
Number of dancers	Rhonda	25
Number of dancers	The dancers	41

A Child's Travel Time Example

Here is another illustration of this third Flaw of Averages that you could present to your class:

One day, a child walks to a friend's home to go play. Going there, she walks the *x* kilometers in a leisurely *t* minutes. Later, the child realizes that she is late for dinner! She decides to run back home, traveling the *x* kilometers in *t/2* minutes.

What is the average speed of the child with respect to her total travel time?

What is the average speed of the child with respect to the distance she traveled?

Do you get the same value for her average speed from these two different perspectives?

The key difference is that the distance traveled is the same in both going to the friend's home and returning to her home, but the time spent traveling in the two cases is different. Hence, the average speed is different from these two different perspectives, as illustrated in detail here:

The average speed with respect to travel time is:

$$\frac{t}{t+t/2} * \frac{x}{t} + \frac{t/2}{t+t/2} * \frac{x}{t/2} = \frac{2}{3} * \frac{x}{t} + \frac{1}{3} * \frac{x}{t/2} = \frac{4}{3} * \frac{x}{t}$$

The average speed with respect to distance traveled is

$$\frac{x}{x+x} * \frac{x}{t} + \frac{x}{x+x} * \frac{x}{t/2} = \frac{1}{2} * \frac{x}{t} + \frac{1}{2} * \frac{x}{t/2} = \frac{3}{2} * \frac{x}{t}$$

Using our table structure illustrates the difference in the outcomes again:

The Average…	**From the Perspective of…**	**Yields a Value of…**
Speed of travel	Time of travel	$\frac{4}{3} * \frac{x}{t}$
Speed of travel	Distance of travel	$\frac{3}{2} * \frac{x}{t}$

Are one of these answers correct and the other incorrect? Not necessarily, as each answers a different question.

However, when someone asks 'what was your average speed', the calculation that that person is likely expecting is 'what was the total distance you traveled divided by the total amount of time that you were traveling?' For this scenario, we get

$$\frac{x+x}{t+{}^{t}\!/_{2}} = \frac{2x}{{}^{3t}\!/_{2}} = \frac{4}{3}*\frac{x}{t},$$

which is the same as what we calculated for the average speed of travel from the perspective of the time of travel. This is often the expected response when asked 'what was your average speed'.

Numerical Instances of the Child's Travel Time Example

Here we will do the Child's Travel Time Example with a few numbers.

<u>Instance #1:</u> The child walks 1 kilometer to a friend's home in 12 minutes. Later, she decides to run back home, traveling the kilometer in 6 minutes.

The average speed with respect to travel time is:

$$\frac{12}{12+6}*\frac{1}{12} + \frac{6}{12+6}*\frac{1}{6} = \frac{2}{3}*\frac{1}{12} + \frac{1}{3}*\frac{1}{6} = \frac{1}{9}\ \frac{\text{km}}{\text{min}} = 6\frac{2}{3}\ \frac{\text{km}}{\text{hour}}$$

The average speed with respect to distance traveled is

$$\frac{1}{1+1}*\frac{1}{12} + \frac{1}{1+1}*\frac{1}{6} = \frac{1}{2}*\frac{1}{12} + \frac{1}{2}*\frac{1}{6} = \frac{1}{8}\ \frac{\text{km}}{\text{min}} = 7\frac{1}{2}\ \frac{\text{km}}{\text{hour}}$$

Using our table structure illustrates the difference in the outcomes for **Instance #1**:

<u>The Average…</u>	**<u>From the Perspective of…</u>**	**<u>Yields a Value of…</u>**
Speed of travel	Time of travel	$6\frac{2}{3}\ \frac{\text{km}}{\text{hour}}$
Speed of travel	Distance of travel	$7\frac{1}{2}\ \frac{\text{km}}{\text{hour}}$

Instance #2: The child walks 3 kilometers to a friend's home in 60 minutes. Later, she decides to run back home, traveling the 3 kilometers in 30 minutes.

The average speed with respect to travel time is:

$$\frac{60}{60+30}*\frac{3}{60} + \frac{30}{60+30}*\frac{3}{30} = \frac{2}{3}*\frac{3}{60} + \frac{1}{3}*\frac{3}{30} = \frac{1}{15}\ \frac{\text{km}}{\text{min}} = 4\ \frac{\text{km}}{\text{hour}}$$

The average speed with respect to distance traveled is

$$\frac{3}{3+3}*\frac{3}{60} + \frac{3}{3+3}*\frac{3}{30} = \frac{1}{2}*\frac{3}{60} + \frac{1}{2}*\frac{3}{30} = \frac{3}{40}\ \frac{\text{km}}{\text{min}} = 4\frac{1}{2}\ \frac{\text{km}}{\text{hour}}$$

Using our table structure illustrates the difference in the outcomes for **Instance #2**:

The Average…	**From the Perspective of…**	**Yields a Value of…**
Speed of travel	Time of travel	$4\ \frac{\text{km}}{\text{hour}}$
Speed of travel	Distance of travel	$4\frac{1}{2}\ \frac{\text{km}}{\text{hour}}$

Instance #3, with a variation on the travel time: The child walks 2 kilometers to a friend's home, taking her 40 minutes. Later, she decides to run home, traveling the 2 kilometers in 15 minutes.

The average speed with respect to travel time is:

$$\frac{40}{40+15}*\frac{2}{40} + \frac{15}{40+15}*\frac{2}{15} = \frac{8}{11}*\frac{2}{40} + \frac{3}{11}*\frac{2}{15} = \frac{4}{55}\ \frac{\text{km}}{\text{min}} = 4\frac{4}{11}\ \frac{\text{km}}{\text{hour}}$$

The average speed with respect to distance traveled is

$$\frac{2}{2+2}*\frac{2}{40} + \frac{2}{2+2}*\frac{2}{15} = \frac{1}{2}*\frac{2}{40} + \frac{1}{2}*\frac{2}{15} = \frac{11}{120}\ \frac{\text{km}}{\text{min}} = 5\frac{1}{2}\ \frac{\text{km}}{\text{hour}}$$

Using our table structure illustrates the difference in the outcomes for **Instance #3**:

The Average…	**From the Perspective of…**	**Yields a Value of…**
Speed of travel	Time of travel	$4\frac{4}{11}\ \frac{\text{km}}{\text{hour}}$
Speed of travel	Distance of travel	$5\frac{1}{2}\ \frac{\text{km}}{\text{hour}}$

Instance #4, with a variation on the distance and time of travel: The child walks a scenic path for 3 kilometers to a friend's home, taking her 60 minutes. Later, she decides to run home following a more direct path, traveling 2 kilometers in 15 minutes.

The average speed with respect to travel time is:

$$\frac{60}{60+15}*\frac{3}{60} + \frac{15}{60+15}*\frac{2}{15} = \frac{4}{5}*\frac{3}{60} + \frac{1}{5}*\frac{2}{15} = \frac{1}{15}\ \frac{\text{km}}{\text{min}} = 4\ \frac{\text{km}}{\text{hour}}$$

The average speed with respect to distance traveled is

$$\frac{3}{3+2}*\frac{3}{60} + \frac{2}{3+2}*\frac{2}{15} = \frac{3}{5}*\frac{3}{60} + \frac{2}{5}*\frac{2}{15} = \frac{1}{12}\ \frac{\text{km}}{\text{min}} = 5\ \frac{\text{km}}{\text{hour}}$$

Using our table structure illustrates the difference in the outcomes for **Instance #4**:

The Average…	**From the Perspective of…**	**Yields a Value of…**
Speed of travel	Time of travel	$4\ \frac{\text{km}}{\text{hour}}$
Speed of travel	Distance of travel	$5\ \frac{\text{km}}{\text{hour}}$

Jogging Example

The following Jogging example is similar to the Child's Travel Time example, with an illustration of a common mistake when working with averages.

Let's assume that I like to walk or jog around a 4 km loop in the morning with the following paces for walking and jogging:

My **walking** speed is 5 km/h

My **jogging** speed is 10 km/h

Using the equation of

$$\text{time} \;=\; \frac{\text{distance}}{\text{speed}}$$

we calculate that when walking around the 4 km loop, it takes me

$$\frac{4\text{ km}}{5\text{ km/h}} \;=\; 0.8\text{ hours} \;=\; 48\text{ minutes}$$

to complete the loop, whereas when jogging around the 4 km loop, it takes me

$$\frac{4\text{ km}}{10\text{ km/h}} \;=\; 0.4\text{ hours} \;=\; 24\text{ minutes}.$$

Now let's consider a day when I decide to walk the first half of the loop and jog the second half of the loop. The question is how long will it take me to complete the loop on this day?

One approach to solving this problem would be to use the following logic:

i) When walking, I complete the loop in 48 minutes, so on this day I will complete the first half of the loop in 24 minutes.

ii) When jogging, I complete the loop in 24 minutes, so on this day I will complete the second half of the loop in 12 minutes.

iii) I will complete the entire loop on this day in 36 minutes.

The 36 minutes is indeed the correct answer.

However, there is another approach that may seem intuitively correct that yields an incorrect solution:

i) If I walk for half of the loop and jog for half of the loop, my average speed with respect to the distance I travel is

$$\frac{5 \text{ km/h} + 10 \text{ km/h}}{2} = 7.5 \text{ km/h}$$

ii) Using the equation of $\text{time} = \dfrac{\text{distance}}{\text{speed}}$, we then calculate that

$$\frac{4 \text{ km}}{7.5 \text{ km/h}} = 0.5\overline{3} \text{ hours} = 32 \text{ minutes}$$

Why did the first approach have a solution of 36 minutes whereas the second approach has a solution of 32 minutes? The discrepancy comes from the question that was asked and the way in which the average speed was calculated in part (i) of the second approach. Since the question was ‘how long will it take me to complete the loop’, we need to use the average speed from the perspective of time of travel around the loop. In the steps above, 7.5 km/h is instead the average speed from the perspective of the distance of the loop.

The main point of this jogging example is that if the average speed is calculated from the wrong perspective, then

$$E[\text{time}] \neq E\left[\frac{\text{distance}}{\text{speed}}\right],$$

which for a fixed distance is equivalently

$$E[\text{time}] \neq \frac{\text{distance}}{E[\text{speed}]}.$$