Supplemental Activities for the Fractals Blended Learning Module

MIT LINC BLOSSOMS

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Extending the Sierpinski triangle

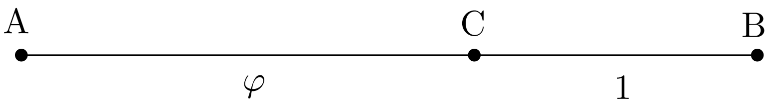
What happens if you don't start the chaos game by choosing one of the vertices, but pick an arbitrary point instead? The resulting picture will still *look* like the Sierpinski triangle, but in general, none of its points will be a Sierpinski triangle point! In fact, each new point that you draw will get closer and closer to the set of points on the Sierpinski triangle, but will never actually reach the set. We say that the set of Sierpinski triangle points is an *attracting* set for the random procedure.

The chaos game on the triangle is just a special case of an beautiful method of generating patterns called *iterated function systems*. These systems are built by defining a number n of affine transformations of points in the plane, and assigning a probability to each of the n (so that all of the probabilities sum to 1). Starting from a single "current" point, choose one of the n transformations according to its probability, and apply the transformation to the current point, drawing the result as a second point in the plane. This second point becomes the new current point, and the process is repeated (randomly choosing a transformation and applying it to generate another point). Iterated function systems are capable of producing beautiful images from simple rules (such as the fern below), and would be an excellent independent study project for a motivated algebra student.

Figure 1: Image courtesy of David Tran, `http://davidtran.doublegifts.com/blog/`.

The Fibonacci numbers and the golden ratio

In his *Elements*, Euclid described the division of a line "in mean and extreme ratio," by which he meant finding the location of a point C on a line AB such that the ratio of the length of CB to the length of AC is the same as the ratio of the length of AC to the length of AB.



If we fix the length of the line segment CB as 1, then the length of AC is denoted by φ. Our requirement on the ratios of line segments can be expressed as:

$$\frac{1}{\varphi} = \frac{\varphi}{\varphi + 1}$$

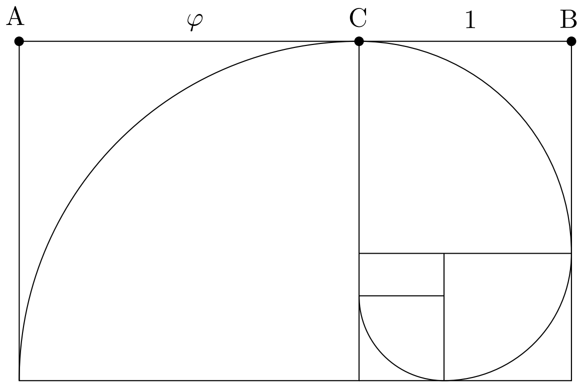
Rearranging gives the following quadratic equation:

$$\varphi^2 - \varphi - 1 = 0.$$

Using the quadratic formula to find the roots of this quadratic expression gives us one positive (φ) and one negative ($\overline{\varphi}$) solution:

$$\varphi, \overline{\varphi} = \frac{1 \pm \sqrt{5}}{2}$$

The positive solution φ is referred to as the *golden ratio* and occurs in many places in nature and art. For example, *golden spirals* often appear in natural phenomena, and can be derived from the golden ratio as shown in the following figure.



There are many interesting algebraic and geometric observations to be made here, but one of the simplest and most remarkable is credited to the eighteenth-century mathematician Abraham de Moivre: we can write the nth Fibonacci number F_n as

$$\frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}.$$

It is not difficult to prove this result by induction. Observe that the definition of F_n requires knowledge of F_{n-1} and F_{n-2}. Thus, for a proof by induction, you need two base cases: F_0 and F_1. Then, for the inductive step, assume that the result holds for F_{n-1} and F_{n-2}, and demonstrate that it must hold for F_n. Hint: remember that $x = \varphi$ and $x = \overline{\varphi}$ both satisfy $x^2 - x - 1 = 0$.

$$F_0 = \frac{\varphi^0 - \overline{\varphi}^0}{\sqrt{5}} = 0$$

$$F_0 = \frac{\varphi^1 - \overline{\varphi}^1}{\sqrt{5}} = \frac{1 + \sqrt{5} - (1 - \sqrt{5})}{2\sqrt{5}} = 1$$

$$
\begin{aligned}
F_n &= F_{n-1}+F_{n-2}=\frac{\varphi^{n-1}-\overline{\varphi}^{n-1}}{\sqrt{5}}+\frac{\varphi^{n-2}-\overline{\varphi}^{n-2}}{\sqrt{5}} \\
&= \frac{\varphi^{n-1}+\varphi^{n-2}}{\sqrt{5}}-\left(\frac{\overline{\varphi}^{n-1}+\overline{\varphi}^{n-2}}{\sqrt{5}}\right) \\
&= \varphi^{n-2}\frac{\varphi+1}{\sqrt{5}}-\overline{\varphi}^{n-2}\left(\frac{\overline{\varphi}+1}{\sqrt{5}}\right) \\
&= \varphi^{n-2}\frac{\varphi^2}{\sqrt{5}}-\overline{\varphi}^{n-2}\left(\frac{\overline{\varphi}^2}{\sqrt{5}}\right) \\
&= \frac{\varphi^n-\overline{\varphi}^n}{\sqrt{5}}
\end{aligned}
$$

More fractal lessons for the classroom

For hundreds of examples of the appearance of fractals in our universe (including art, literature, biology, geology, finance and music), see Michael Frame and Benoit Mandelbrot's website *A Panorama of Fractals and their Uses* `http://classes.yale.edu/Fractals/Panorama/`. There are lots of excellent teaching materials for introducing middle- and high-school students to fractal mathematics. In particular, I recommend:

Jonathan Choate, Robert Devaney, Alice Foster, Iteration: A Toolkit of Dynamics Activities, Key Curriculum Press, 1998.

Jonathan Choate, Robert Devaney, Alice Foster, Fractals: A Toolkit of Dynamics Activities, Key Curriculum Press, 1998.

Robert Devaney, The Mandelbrot and Julia sets: A Toolkit of Dynamics Activities, Key Curriculum Press, 1999.

Heinz-Otto Peitgen, Hartmut Jurgens, and Dietmar Saupe, Fractals for the Classroom, Parts I and II, Springer, 1992.

Michael Frame and Benoit Mandelbrot, Fractals, graphics, and mathematics education, Mathematical Association of America, 2002.

Chaos game script

This script was written in the MATLAB programming language.

```
clear all, close all

% Drawing a triangle.

h = figure;
set(h,'Color',[1 1 1])
xy = [0 10 5; 0 0 sqrt(75)];
line([0;10],[0;0],'LineWidth',2,'Color','k')
line([0;5],[0;sqrt(75)],'LineWidth',2,'Color','k')
line([5;10],[sqrt(75);0],'LineWidth',2,'Color','k')
axis square, axis off, hold on

p = xy(:,1);
eqvec = [1/3,2/3,1];
corner = [];
plot(p(1),p(2),'r*')
for i = 1:6000
    corner = min(find(eqvec>rand));
    if corner == 1
        str = 'r*';
    elseif corner == 2
        str = 'b*';
    else
        str = 'g*';
    end
    p = (p+xy(:,corner))/2;
    plot(p(1),p(2),str)
    pause(1)
end
```