Teacher's Guide for the Fractals Blended Learning Module

MIT LINC BLOSSOMS

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Preparation and implementation

This lesson requires a knowledge of basic algebra, plus the mechanics of complex arithmetic. A tutorial on complex arithmetic accompanies this lesson, for students to complete the night in advance (if this material is not a part of your curriculum). We have also prepared a quick quiz on this material that you could work through as a class to refresh everyone's memory (or have students take individually to provide extra motivation to complete the tutorial!). The lesson begins with a thought experiment/demonstration that can be implemented in several ways. Feel free to choose the method that will work best in your classroom.

Part 1: Dots, dots and more dots

In this part, we introduce the chaos game and ask students to speculate on the resulting pattern of dots. This activity is taken from Robert Devaney's Chaos in the Classroom.[1] In our classroom, we ran this activity as a thought experiment, allowing students to brainstorm in pairs and try drawing a few points on their own before reconvening and assembling their guesses. This worked very well! Record all of the students observations somewhere and save them for the end of lecture.

If you'd like to try generating the points in class, Devaney suggests a method using an overhead projector, markers and transparency sheets. Give each student (or pair of students) an overhead transparency sheet with the labeled triangle already drawn on it, and something he or she can use to generate R, G and B (such as a single die, or drawing marbles from a bag). Using a permanent marker and a ruler, have each student draw as many points (carefully!) as he or she can within some short time window. Once the time limit is up, collect all of the sheets, stack them on top of each other, and project the result. A word of warning: when we tried this, the result was a messy disaster that had no clear pattern whatsoever!

The video ends with a computer animation of one run of the chaos game. If you have access to a computer and would rather run your own simulation in class, then you can program the procedure to run as quickly and as many times as you like. You could get it started at the beginning of class, then put it away and bring it out again at the end.

For interesting extensions to the Sierpinski triangle, see the Supplemental Notes.

Part 2: Difference equations

This section uses the Fibonacci numbers to introduce difference equations. These numbers are a very ripe topic for high school algebra and geometry classes because of their connections to the *golden ratio*, denoted by φ. For more activities related to the golden ratio, see the Supplemental Notes.

At the end of this part, the class is asked to explore the behavior of several difference equations. It works well to have students work in groups to explore the patterns, then to hold a large class discussion to combine observations.

1. $x_{n+1} = -\frac{1}{2}x_n - \frac{3}{2}$ *Result will converge to* -1 *in a alternating manner.*
2. $x_{n+1} = x_n^2$ *Result will converge or diverge straightaway depending on whether* $|x_0| < 1$.

Part 3: Bounded trajectories

This section suggests categorizing trajectories as either bounded or unbounded, which will be the key idea in the next section on the Mandelbrot set. The mathematical criterion for boundedness can be quite abstract for high school students, so it's worth taking time to make sure that your class understands it before moving on. In my classroom, I found it helpful to frame the idea this way: if a trajectory is unbounded, then it's not possible to find a single number B such that the absolute value of the trajectory is always less than B. If you can find such a B, even if it's really big, then the trajectory is bounded.

[1] http://math.bu.edu/DYSYS/chaos-game/chaos-game.html

Working with one of the examples is also useful. For example, if a student argues that a particular trajectory is unbounded, then ask them to find the point in the trajectory whose absolute value will exceed a particular value, say 10000. If a trajectory is bounded, ask them to find a particular number B such that the absolute value of the trajectory is always less than B. An issue that often confuses students is the non-uniqueness of B in a bounded trajectory. If the absolute value is always less than 2, then it is also always less than 3! The definition of boundedness only requires the *existence* of such a B, not the identification of a single unique value.

To practice with this concept, the students are asked to look at the trajectories they drew earlier and classify them as bounded or unbounded.

Part 4: A complex difference equation

Here, we revisit one of the difference equations that we practiced with earlier, but add a twist: we allow the equation to have an additive complex parameter c. To be able to assess the boundedness of a complex trajectory, the students will have to be able to find to absolute value of a complex number (a skill that is practiced in the complex arithmetic tutorial that accompanies this lecture). As in the problems in the previous section, it works well to have the students attempt the problems in small groups, then convene to get a final class answer before moving on.

B. If $c = 1$, what are x_1, x_2, x_3 and x_4? $1, 2, 5, 26$. *An unbounded trajectory.*

C. If $c = i$ (where i is the imaginary number), what are x_1, x_2, x_3 and x_4? $x_1 = i, x_2 = -1+i, x_3 = -i, x_4 = -1+i$. *Another bounded trajectory (and one that repeats every four steps). This is a great example of a* periodic *trajectory, one that repeats itself!*

Part 5: The Mandelbrot set

Being able to successfully apply the coloring scheme requires that the students feel comfortable with looking at the complex plane and finding different points in it. Don't worry about spending too much time on getting guesses from the class on this one - the answer is incredibly surprising!

Part 6: Fractals

This section depicts the Mandelbrot set, and includes a computer-generated "zoom" into a portion of the set to show its infinite complexity. We also discuss the presence of fractal patterns in nature, and their use in computer-generated artwork.

We finish the lesson by discussing the Sierpinski triangle and coming back to the chaos game, including an animation of a trial run of the game. This is an incredibly fascinating example, which I hope encourages students to pursue the topic on their own!

If you'd like some more ways to incorporate fractal mathematics into your classroom, see the Supplemental Notes.